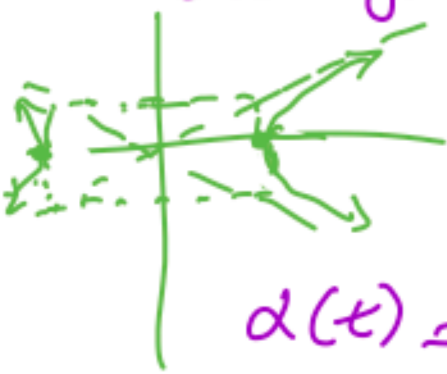
$z^2 + y^2 = x^2 + 4$ ⟵ sections mean e.g. set $x=2$

Another example

Parametrize $x^2 - 4y^2 = 1$

One way. let $y = t$



$x^2 - 4t^2 = 1$

$x^2 = 1 + 4t^2$

$x = \pm\sqrt{1+4t^2}$

$\alpha(t) = \left(\sqrt{1+4t^2}, t\right) : t \in \mathbb{R}$

$\beta(t) = \left(-\sqrt{1+4t^2}, t\right) : t \in \mathbb{R}$.

Another way: use hyperbolic trig fcns

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

If we let $e^{ix} = \cos(x) + i\sin(x)$

$e^x = \cosh x + \sinh x$

Identities ~ similar to $\cos(x)$ & $\sin(x)$ identities.

$$\cosh^2(x) - \sinh^2(x) = 1$$

[Why? $\left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$

$$= \frac{1}{4}\left(e^{2x}+e^{-2x}+2\underbrace{e^x e^{-x}}_{1}\right) - \frac{1}{4}\left(e^{2x}+e^{-2x}-2\underbrace{e^x e^{-x}}_{1}\right)$$

$$e^x e^{-x} = 1$$

$$= \cancel{\frac{1}{4}e^{2x}} + \cancel{\frac{1}{4}e^{-2x}} + \frac{2}{4} - \cancel{\frac{1}{4}e^{2x}} - \cancel{\frac{1}{4}e^{-2x}} + \frac{2}{4}$$

$$= \boxed{1}.$$]

Using this: To parametrize

$$x^2 - y^2 = 1$$

Let $\boxed{\begin{array}{l} x = \cosh(t) \\ y = \sinh(t) \end{array}}$

$\cosh^2(t) - \sinh^2(t) = 1$ ✓

Back to our example $x^2 - 4y^2 = 1$

Let $\left.\begin{array}{l} x = \cosh(t) \\ y = \frac{1}{2}\sinh(t) \end{array}\right\}$ right half

Plug in: $\cosh^2(t) - 4\left(\frac{1}{2}\sinh(t)\right)^2$

$= \cosh^2(t) - \sinh^2(t) = 1$ ✓

Other half:

$x = -\cosh(t)$

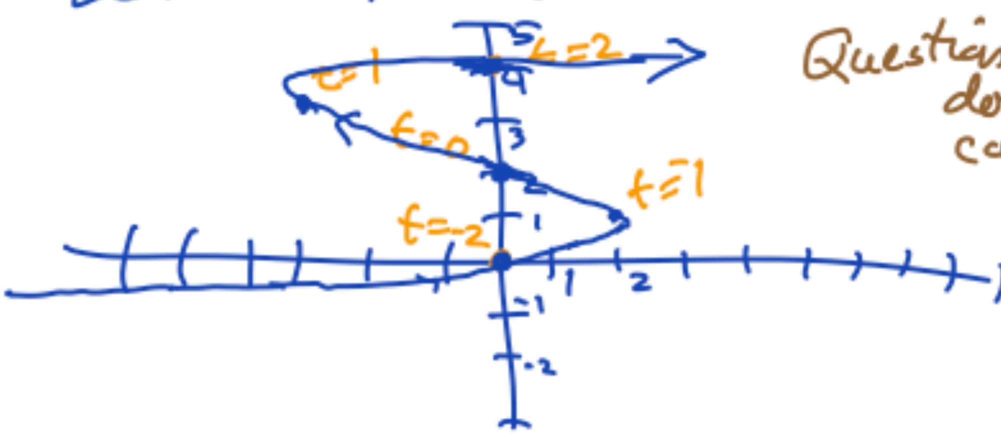
$y = \frac{1}{2}\sinh(t)$

Other fun things about cosh & sinh:

$$\begin{cases} (\cosh(t))' = \sinh(t). \\ (\sinh(t))' = \cosh(t). \end{cases}$$

Other aspects of curves

Let $\alpha(t) = (t^3 - 4t, t + 2)$



Question: where does this curve turn?

Velocity vector: $\alpha'(t) = (3t^2-4,\ 1)$

This is vertical when $3t^2-4=0$

$$\Rightarrow t^2 = \frac{4}{3} \Rightarrow t = \pm\sqrt{\frac{4}{3}}.$$

Question: when is the particle moving along the curve slowest? fastest?

$$\text{Speed} = \|\alpha'(t)\| = \sqrt{(3t^2-4)^2+1}$$

This number is smallest when $3t^2-4=0$, i.e. again $t=\pm\sqrt{\frac{4}{3}}$.

speed $= 1$ unit/unit time.

For very large positive or negative t, it could be really large.

The unit tangent vector $T(t)$ of the curve is the "normalized" velocity vector. (v is normalized as $\frac{1}{\|v\|}v$, so then it has length 1.)

In our example,

$$\alpha'(t) = (3t^2-4,1)$$

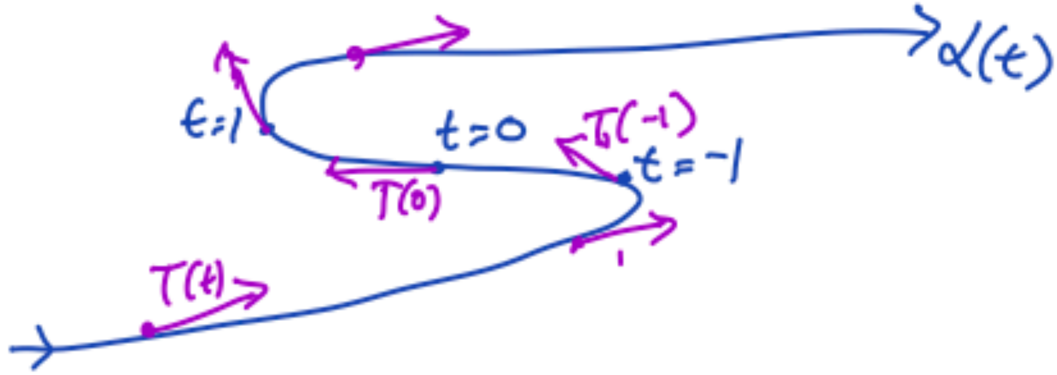
$$T(t) = \frac{1}{\|\alpha'\|}\alpha' = \frac{1}{\sqrt{(3t^2-4)^2+1}}(3t^2-4,1)$$

$$T(t) = \frac{1}{\sqrt{(3t^2-4)^2+1}}\left(3t^2-4,\, 1\right).$$

Notice for any vector v,

$$\left\| \underbrace{\frac{1}{\|v\|}}_{c} v \right\| = |c|\,\|v\| = \frac{1}{\|v\|}\cdot\|v\| = 1.$$

really is a unit vector now.



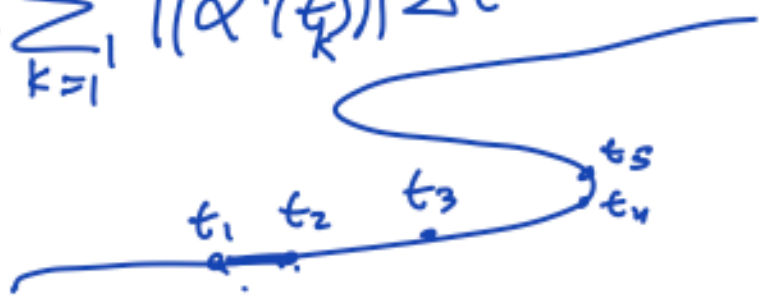
You can use the unit tangent vector to measure the curvature of the curve.

To compute the arclength of a curve.

Idea: (speed)(time) = (distance)

Total distance traveled along $\alpha(t)$ between $t=a$ & $t=b$

$$= \lim_{\Delta t \to 0} \sum_{k=1}^{n} \|\alpha'(t_k)\| \Delta t$$



This looks like a limit of Riemann sums, i.e. integral.

$$\Rightarrow \begin{pmatrix} \text{arclength} \\ \text{along } \alpha \text{ from} \\ t=a \text{ to } t=b \end{pmatrix} = \boxed{\int_a^b \|\alpha'(t)\| \, dt}$$

In our example, the length of $\alpha(t) = (t^3-4t, t+2)$ between $t=-1$ and $t=1$ is

$$s(t) = \int_{-1}^{1} \sqrt{(3t^2-4)^2+1} \, dt \approx$$

$\approx \ldots$ numerical calculation.

$s'(t) = \|\alpha'(t)\| = \text{speed}.$

<u>Curvature</u> Given a curve $\alpha(t)$, we can find its velocity vector $\alpha'(t)$ & unit tangent vector $\frac{1}{\|\alpha'(t)\|}\alpha'(t) = T(t)$.

The <u>curvature vector</u> at time t is $\frac{dT}{ds}$ = (derivative of the unit tangent vector with respect to arclength)

$$\frac{d}{ds}T(s) = \frac{dT/dt}{ds/dt} = \frac{T'(t)}{s'(t)}$$

$$\Rightarrow \boxed{\frac{dT}{ds} = \frac{1}{\|\alpha'(t)\|}T'(t) = K(t)}$$

Curvature vector at time t.

Sometimes, we just want the "curvature", which is $\|K(t)\|$.

For our example,

$$\alpha(t) = (t^3 - 4t,\ t+2)$$

$$T(t) = \frac{1}{\sqrt{(3t^2-4)^2+1}}\left(3t^2-4,\ 1\right)$$

$$= \left(\frac{3t^2-4}{\sqrt{(3t^2-4)^2+1}},\ \frac{1}{\sqrt{(3t^2-4)^2+1}} \right)$$

$$T'(t) = \left(\text{yuck}_1(t),\ \text{yuck}_2(t) \right)$$

using quotient rule & chainrule.

$$K(t) = \frac{1}{\sqrt{(3t^2-4)^2+1}}\left(\text{yuck}_1(t),\ \text{yuck}_2(t) \right)$$

$$\|K(t)\| = \frac{1}{\sqrt{(3t^2-4)^2+1}} \sqrt{(\text{yuck}_1(t))^2 + (\text{yuck}_2(t))^2}.$$

Surfaces

example ① $z = xy^2 - y$ ⟵ gives vertical position at each (x,y) in the plane.

② $z = 4x^2 - y^2$.

Particular examples of functions $f(x,y)$

$z = f(x,y)$ ⟵ graph of function.

Derivatives of functions of several variables

partial derivative ⟵ Take deriv. with respect to one variable while holding other variables constant.

domain $\mathbb{R}^1$ $\quad f(x) \Rightarrow f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

"$\dfrac{df}{dx}$"

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domain $\mathbb{R}^2$ $\quad f(x,y) \Rightarrow \dfrac{\partial f}{\partial x} = \lim_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$
